

VIOLATION OF CAUSALITY IN $f(R)$ GRAVITY

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We examine the question as to whether the $f(R)$ gravity theories, in both metric and in Palatini formalisms, permit space-times in which the causality is violated. We show that the field equations of these $f(R)$ gravity theories admit solutions with violation of causality for a physically well-motivated perfect-fluid matter content.

1. Introduction

Generalization of the Einstein-Hilbert Lagrangian by replacing the scalar curvature R by a general function $f(R)$, gives rise to theories known as $f(R)$ gravity. These theories have recently received considerable attention (see, e.g., Refs. 1 and references therein) motivated mainly by the fact that they can explain the observed accelerating late expansion of the universe with no dark energy component. They also offer a much richer framework than General Relativity (GR) in the sense that $f(R)$ theories has two different formulations known as metric and Palatini formalisms, which give rise to very different equations of motion. In the metric formulation it is assumed that the metric and connections are compatible, while in the Palatini formulation the equations of motion are obtained by considering the metric and connection as independent fields in the variation of the action. Much efforts from the theoretical viewpoint have been developed so far in order to clarify inherent subtleties of these theories, including general principles such as the so-called energy conditions.² Another interesting issue is the question as to whether these theories permit space-time solutions in which causality is violated. It is well-known that in GR there are solutions to the field equations that possess causal anomalies in the form of closed time-like curves. The solution found by Gödel³ is the best known example. In this context, we have recently examined the violation of causality in both formulations of $f(R)$ gravity.⁴ Here we extend the previous works by presenting an unified view of causality problem in both versions $f(R)$ gravity.

2. Metric and the Palatini Approaches

The action that defines an $f(R)$ gravity is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{f(R)}{2\kappa^2} + \mathcal{L}_m \right], \quad (1)$$

where $\kappa^2 \equiv 8\pi G$, g is the determinant of the metric $g_{\mu\nu}$ and \mathcal{L}_m is the Lagrangian density for the matter fields. Varying this action with respect to the metric we obtain the field equations

$$f_R R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) f_R = \kappa^2 T_{\mu\nu}, \quad (2)$$

where $f_R \equiv df/dR$, $\square = g^{\alpha\beta} \nabla_\alpha \nabla_\beta$, and $T_{\mu\nu}$ is the energy-momentum tensor.

In the Palatini formulation we treat the metric and the connection as independent fields, and variation of the action (1) gives the field equations

$$f_R R_{(\mu\nu)} - \frac{f}{2} g_{\mu\nu} = \kappa^2 T_{\mu\nu}. \quad (3)$$

Here the Ricci tensor $R_{\mu\nu}$ must be calculated using the connections field given by

$$\Gamma_{\mu\nu}^\rho = \{\rho_{\mu\nu}\} + \frac{1}{2f_R} (\delta_\mu^\rho \partial_\nu + \delta_\nu^\rho \partial_\mu - g_{\mu\nu} g^{\rho\sigma} \partial_\sigma) f_R, \quad (4)$$

where $\{\rho_{\mu\nu}\}$ are the Levi-Civita connections of the metric $g_{\mu\nu}$.

3. Gödel-type Geometry

The homogeneous Gödel-type metric in cylindrical coordinates (r, ϕ, z) is given by⁶

$$ds^2 = dt^2 + 2H(r) dt d\phi - dr^2 - G(r) d\phi^2 - dz^2, \quad (5)$$

where $G(r) \equiv D^2 - H^2$, $H(r) = (4\omega/m^2) \sinh^2(mr/2)$ and $D(r) = \sinh(mr)/m$, with ω and m being constant parameters such that $\omega^2 > 0$ and $-\infty \leq m^2 \leq +\infty$. The Gödel solution³ is a particular case in which $m^2 = 2\omega^2$. The existence of closed time-like curves of Gödel-type depends on the behavior of the function $G(r)$. If $G(r) < 0$ for a certain range of r ($r_1 < r < r_2$, say) *Gödel circles* defined by $t, z, r = \text{const}$ are closed time-like curves. The causality features of the Gödel-type spacetimes thus depend on the two independent parameters m and ω . For $m = 0$ there is a critical radius, defined by $\omega r_c = 1$, such that for all $r > r_c$ there are noncausal Gödel circles. For $0 < m^2 < 4\omega^2$ noncausal Gödel circles occur for $r > r_c$ such that $\sinh^2(mr_c/2) = 1/(4\omega^2/m^2 - 1)$. Clearly this type of violation of causality is not of trivial topological nature, which are obtained by topological identification.⁵

4. Non-causality in f(R) Gravity

We can simplify our calculations choosing the tetrad basis $(\theta^0, \theta^1, \theta^2, \theta^3) = (dt + H(r)d\phi, dr, D(r)d\phi, dz)$, relative to which the field equations in both formalism [Eq. (2) and Eq. (3)] reduce to

$$f_R G_{AB} = \kappa^2 T_{AB} - \frac{1}{2} (\kappa^2 T + f) \eta_{AB}, \quad (6)$$

where T is the trace of energy-momentum tensor T_{AB} and we have taken into account that for Gödel-type metrics the Ricci scalar is constant $R = 2(m^2 - \omega^2)$. An important ingredient of the causality problem in Gödel-type universes is the

matter source, which consider a perfect fluid of density ρ and pressure p . Thus, the field equations (6) give us

$$2(3\omega^2 - m^2)f_R + f = \kappa^2(\rho + 3p), \quad (7)$$

$$2\omega^2 f_R - f = \kappa^2(\rho - p), \quad (8)$$

$$2(m^2 - \omega^2)f_R - f = \kappa^2(\rho - p). \quad (9)$$

Equations (8) and (9) give $(m^2 - 2\omega^2)f_R = 0$. Hence, for $f(R)$ theories that satisfy the condition $f_R > 0$, we must have $m^2 = 2\omega^2$ which defines the Gödel metric. The above field equations are then rewritten as

$$\kappa^2 p = f/2 \quad \text{and} \quad \kappa^2 \rho = m^2 f_R - f/2, \quad (10)$$

where f and f_R are valuated at $R = m^2$. This result can be seen as a extension of Bampi and Zordan⁷ GR result to the context of $f(R)$ gravity in both formulation, in the sense that for arbitrary ρ and p (with $p \neq -\rho$) every perfect fluid Gödel-type solution of $f(R)$ gravity, which satisfies the condition $f_R > 0$, is necessarily isometric to the Gödel geometry. Concerning the causality features of these solutions we first note that since they are isometric to Gödel geometry they exhibit noncausal Gödel circles for $r > r_c$. On the other hand, taking into account the above results we have that, in the framework of $f(R)$ gravity, r_c is given by $r_c = 2 \sinh^{-1}(1) \sqrt{2f_R/(2\kappa^2\rho + f)}$, making apparent that the critical radius depends on both the gravity theory and the matter content.

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